JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 28, No. 1, January–February 2005

Geometry-Based Parametric Modeling for Single-Pursuer/ Multiple-Evader Problems

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We examine the situation where a single pursuer must develop an optimal strategy with respect to multiple evaders. The pursuers' objective is to capture an evader, whereas each evader's objective is to avoid capture. This problem is unique in that previous pursuit–evasion problems studied in the literature address one-on-one engagements. Current techniques for solving pursuit–evasion games rely on the construction of trajectories in space–time by applying either a minmax strategy at each time step or through evolutionary/simulation techniques. However, even for one-on-one pursuit–evasion problems, differential game theoretic models with incomplete information result in intractable formulations without saddle point solutions. Evolutionary and simulation-based methods require extensive computational time and resources to arrive at strategies that may not even be provably optimal. By adding multiple evaders to the mix, these "construction-based" methods are severely handicapped. The approach taken in this study is to rapidly develop bounds on the pursuer's capture (evader's escape) possibilities for use in an optimization program. Algorithms are presented for computing isochronal contour maps enclosing regions of achievable dynamics and effective capture (i.e., weapon) performance. In combination, these contour maps form the basis for a parametric model of the engagement, greatly reducing the complexity of the problem.

I. Introduction

THE pursuit–evasion problem is fundamentally an asymmetric differential game. The problem is extremely well studied and has led to many important results. See Refs. 1–3 for an extensive survey and bibliography on differential games for pursuit–evasion problems. Nonetheless, it is also widely recognized that game theory has limitations when the pursuer and/or evader has imperfect information about the other's state, stochastic dynamics, variable velocities, and a non-zero-sum payoff structure. If the relative state and payoff of pursuer and evader cannot be captured in extensive or normal form, as occurs when there may be many codependent strategies, then game theory has little use in practice. As a result, many researchers have applied evolutionary and simulation-based algorithms to these problems, where many simulations runs are made with randomized initial conditions in a attempt to fathom the solution space. Simulation-based solution methods[4] require the pursuer and evader each to make some observation of their relative state and then to choose from some system of primitive strategies an appropriate move. Observations and strategies may be chosen iteratively, within either an event-driven or a discrete-time framework. Simulations often rely on an artificial intelligence technique such as an inference engine, evolutionary algorithm, or neural network. Of course simulation-based methods are ad hoc by their very nature and randomizing the initial conditions does not guarantee that the optimal solution will be enumerated.

An alternative to simulation-based methods is a method of parameterization. The object of parameterization is to develop an optimization program for a simplified game-theoretic formulation of the problem and then introduce the complicating behaviors as constraints. This is essentially the approach of minmax in that the parameters express the worst-case situations, which bound the interaction space. The parameters considered in this paper are induced by the geometry of the problem. The remainder of the paper is organized as follows. In Sec. II, the geometry-based technique is introduced. In Sec. III, the single-pursuer/single-evader case is developed and it is generalized to the single-pursuer/multiple-evader problem in Sec. IV. Concluding remarks are made in Sec. V.

II. Geometry-Based Parameterization

The approach begins by examining the relative geometry of a pursuer and a single evader. We assume that the game is symmetric in that there are two vehicles and either vehicle may be the pursuer or evader. The fundamental characteristics of the problem are observed and then parameterized. Independent parameters of large effect may be chosen as decision variables in an optimization program, whereas dependent parameters can be included as constraints.

For the single-pursuer/single-evader engagement the following situations are possible:

1) The pursuer reaches an advantageous position and can catch the evader for any strategy (maneuver) the evader may present.

2) The pursuer first observes the evader in a position such that the evader can escape capture for any strategy of the pursuer, provided that the evader uses the correct strategy.

3) The pursuer is pursued by the evader (symmetric case). Results of the engagement depend on the initial positions of vehicles, their performance, their capture performance (e.g., weapon performances), and their respective strategies.

The results are fundamentally one of the following:

1) The pursuer catches the evader.

2) The evader catches the pursuer.

3) Both pursuer and evader are captured or destroyed. This is possible simply due to the latent effect of their respective capture processes, for example, the ingress time of a deployed weapon.

4) Neither pursuer nor evader is captured.

We assume that the pursuer is the vehicle that initiates the game by making the first observation of the evader. We can always make this assumption because the opposite case is symmetric. We also assume that the evader has the potential for observing the pursuer

Received 3 September 2003; presented as Paper 2003-6611 at the AIAA 2nd "Unmanned Unlimited" Systems, Technologies, and Operations—Aerospace, Land, Sea Conference and Workshop, San Diego, CA, 15–18 September 2003; revision received 18 March 2004; accepted for publication 22 March 2004.

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immediately upon detection. This will be the initial condition of the problem, which we assume to be uncontrolled because the initial positions of the pursuer and evader are uncontrolled. Thus, if the capture (weapon) performances are approximately equal, the outcome of the encounter depends only on the chosen strategies of the vehicles.

The dynamic limitations for each vehicle may be parameterized as two main effects: a minimum turning radius and a maximum range. If we know the vehicle's maximum dynamic overload, we can easily find the minimum turning radius. Of course a constant turning radius implies a constant velocity; however, as we shall see in Sec. III, the actual speeds are unimportant: the relative speed is the true issue. Hence as long as the vehicle speeds are clearly specified uniformly relative to one another, for example, both maximum, both at cruise speed, and so forth, the result will be realistic. The equation for overload is

$$n = V^2/gR \tag{1}$$

where n is overload, V is speed (m/s), $g = 9.81$ m/s^2 is the Earth's gravity, and R is the turning radius (m). From Eq. (1) the minimum radius is when the overload is at the maximum,

$$R = V^2/gn \tag{2}$$

which is plotted in Figs. 1 and 2. Figure 1 is computed for subsonic vehicles and missiles, whereas Fig. 2 is computed for supersonic vehicles and missiles. As can be seen, the turning radius reaches 2–4.5 km for typical overloads 2–5 g and speeds $V = 300$ m/s for

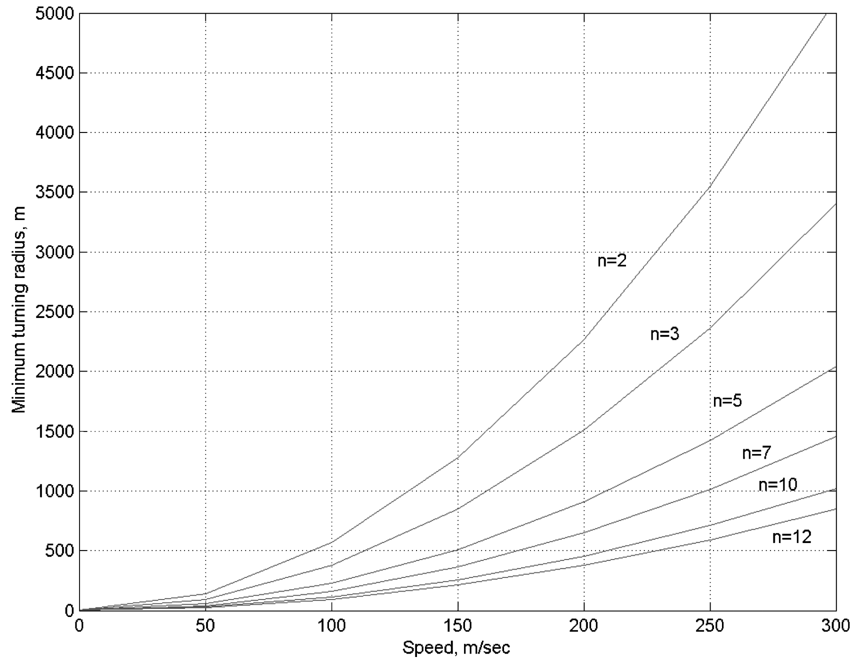

Fig. 1 Minimum turning radius for subsonic vehicles.

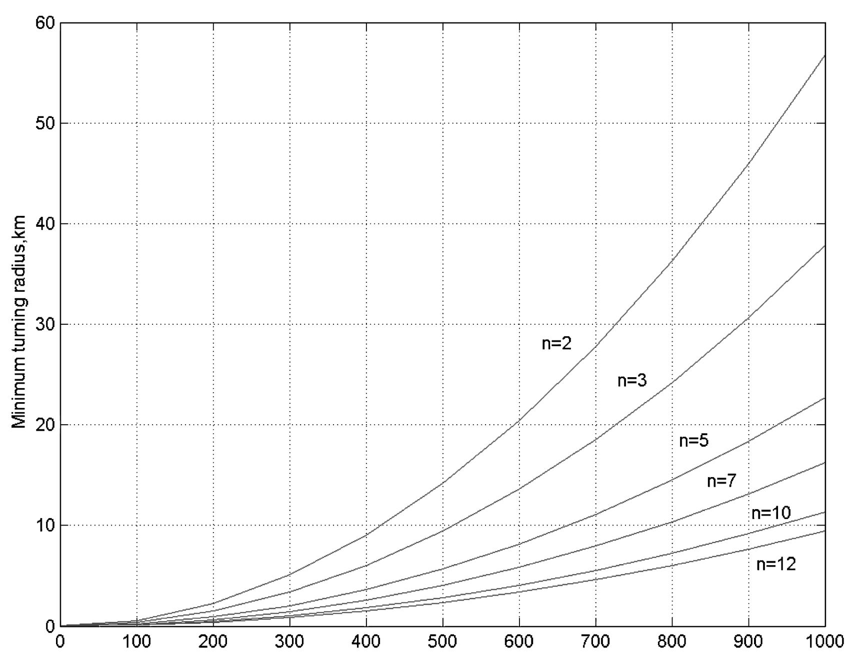

Fig. 2 Minimum turning radius for supersonic vehicles.

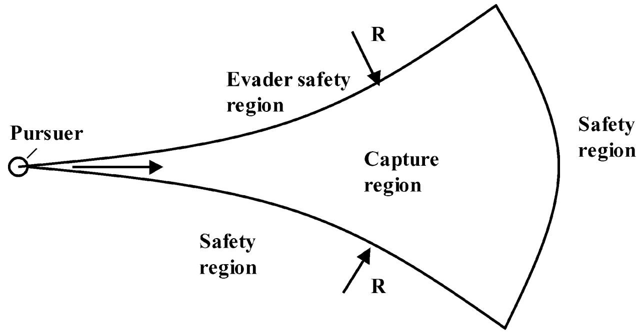

Fig. 3 Maximum-range contour for typical missile.

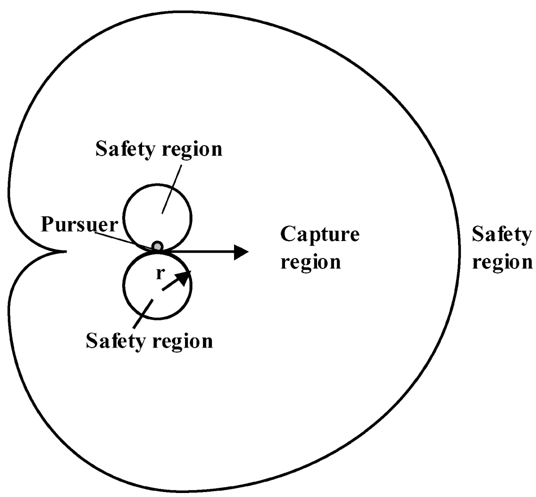

Fig. 4 Maximum-range contour for typical aircraft.

subsonic vehicles. For supersonic vehicles the turning radius may reach 15–20 km for typical overloads of 5–7 g and speeds of about Mach 3.

The maximum range of a vehicle traveling in a straight line is known to be a function of altitude, fuel capacity, lift-to-drag ratio L/D, and so on. A vehicle's maximum range for all possible headings may likewise be determined as a function of turn radius and consequently loss of lift in a bank condition. This maximum-range contour may be normalized to maximum range in a straight line and graphed.

Figure 3 shows a maximum-range contour for a typical high-speed missile in the horizontal plane. Figure 4 shows a typical maximum-range contour for a subsonic aircraft. Notice in each figure that the contour encloses a continuous region that is "reachable" by the pursuer aircraft. Outside the contour is the region in which an evader is "safe." Interesting effects may occur, as in Fig. 4, where there are two circular voids tangent to the initial velocity vector at the aircraft initial position. Of course, given sufficient range (time) these circular voids may vanish.

We now study in detail the possible interactions and movement of the pursuer and evader in the horizontal plane. We may neglect altitude for the moment if we assume that the cruise altitudes are relatively low with respect to straight-line horizontal range or by scheduling the maximum-range contour by altitude.

III. Single-Evader Planar Case

In the single-evader case the pursuer and evader can assume any position and direction/velocity with respect to one another. In this section we introduce a method for viewing, in simple form, the capture and safety regions for this situation. This result will be extended to the multiple evader case.

Return to Figs. 3 and 4 for a moment and add to these graphs an important detail: isochronal contours. If the vehicle speed is constant, these contours are easily created because the distance is proportional to the elapsed time. See Figs. 5 and 6 for examples.

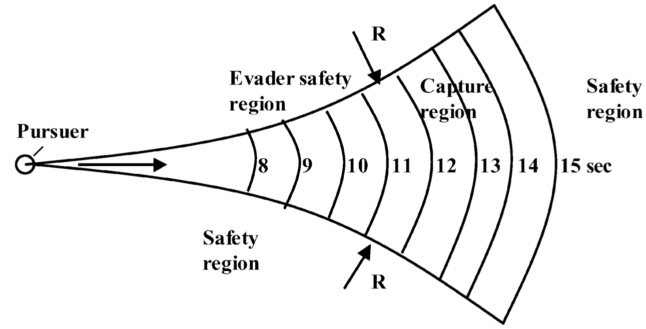

Fig. 5 Capture region with isochronal contours for a missile.

The isochronal contours of Figs. 5 and 6 can be computed. First an open curve set is generated, as illustrated in Fig. 7. Then the closed isochronal curves are constructed from the open curve set. An open curve may be constructed as the coordinates for an equidistant curve in Cartesian space:

$$x = a(\cos\varphi + \varphi\sin\varphi) \tag{3}$$

$$y = a(\sin\varphi - \varphi\cos\varphi) \tag{4}$$

where a is r or R, minimum turning radius of vehicle of pursuer or evader; φ is angle in radians, which is varied over $[0, 2\pi]$; and x, y is point set C in Fig. 7.

We obtain additional curves by turning the system coordinate an angle ψ around the origin 0:

$$X = x\cos\psi + y\sin\psi \tag{5}$$

$$Y = -x\sin\psi + y\cos\psi \tag{6}$$

where ψ is angle of coordinate turn varied over $[0, 2\pi]$ and X and Y are coordinates of the curve. The result is a family of Archimedian spirals with origins on the circle of radius a. The location of the origin on the circumference of the circle is determined by ψ. Using this open curve set, the closed isochronal curves may now be constructed. The idea is to create a symmetry on the X axis at $X = 0$. First we move X to the right by a, $X_1 = X + a$, and consider only $X_1 \geq 0$. For the region $X_1 < 0$, the value $X_1 = -X - a$. In all cases, $Y_1 = Y$. The resulting curves of (X_1, Y_1) are the isochronal contours as plotted in Fig. 6.

The distance *OBC* is

$$D = a(\phi + \psi) \tag{7}$$

To use these isochronal contours, plot the curve sets as generated for each of a pursuer and evader on the same page, as in Fig. 8, with their respective origins aligned with the initial conditions of the problem. In two dimensions, the initial conditions are the location of each vehicle in a common Cartesian frame and the heading angle: the rotation angle of the y axis from the common frame to the vehicle frame. In the vehicle frame, the velocity vector is aligned with the y axis. The evader can escape capture if it has a contour that reaches the safety region, which is less than the time curve of the pursuer. For example, consider the location of the pursuer and evader shown in Fig. 8. The evader can reach the safety region in a minimum time of 6 s (point C), whereas the pursuer can reach point C in a minimum time of 13.9 s. Clearly, in this example the evader escapes, assuming that it uses the best strategy. In Fig. 9 the initial conditions are such that the pursuer "leads" the evader. However, the evader has greater agility and is able to evade capture by turning counterclockwise very early in its trajectory.

IV. Multiple Evaders Case

Assume that there are N vehicles, consisting of 1 pursuer and $N - 1$ evaders. The interaction between each evader and the pursuer

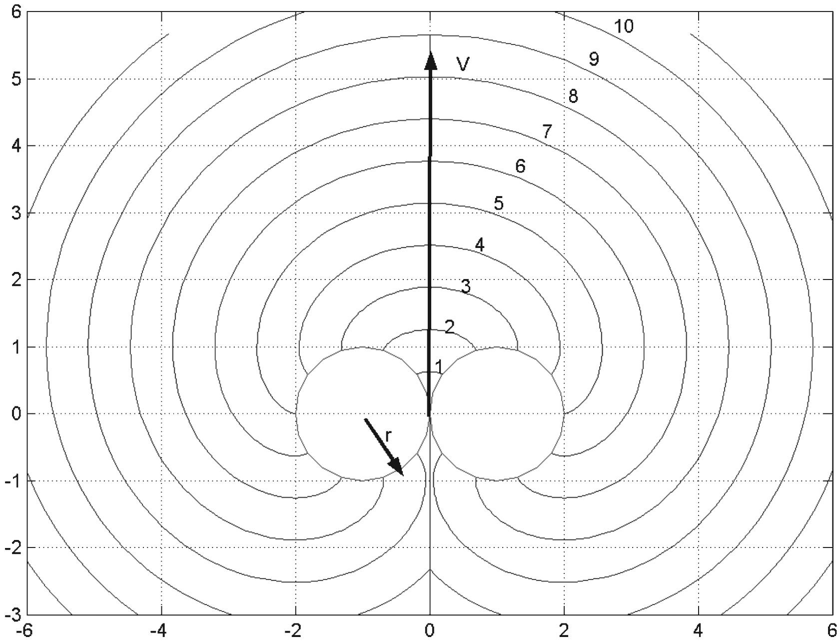

Fig. 6 Capture region with isochronal contours for an aircraft.

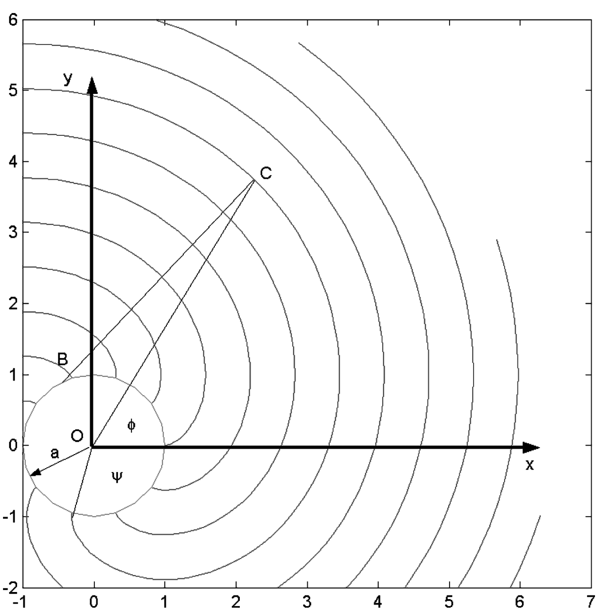

Fig. 7 Construction of equidistant open-curve set.

must now be captured in a system of equations. The equations of motion for all of the vehicles are

$$\frac{dx_i}{dt} = V_i \cos\omega_i \tag{8}$$

$$\frac{dy_i}{dt} = V_i \sin\omega_i \tag{9}$$

$$\frac{dV_i}{dt} = \frac{(T_i - D_i)}{m_i} \tag{10}$$

$$\omega_i = \frac{gn_i}{V_i}, \qquad D_i = \frac{C_{Di}\rho V_i^2 S_i}{2}, \qquad 0 \le T_i \le T_{\max}|n_i| \le n_{\max} \tag{11}$$

where x_i is the ith vehicle coordinate along axis x (m), y_i is the ith vehicle coordinate along axis y (m), t is time (s), V_i is the speed of vehicle i (m/s), ω_i is heading angle between V and axis y of vehicle i (rad), T_i is thrust of vehicle i (N), D_i is drag of vehicle i (N), m_i is mass of vehicle i (kg), $g = 9.81$ m/s^2 is the coefficient of gravity, n_i is overload of vehicle i, C_{Di} is the drag coefficient of vehicle i, ρ is air density (kg/m^3), and S_i is wing area of vehicle i (m^2). Equations (8–11) must be written and computed for each vehicle, $i = 1, 2, \ldots, N$. The isochronal contours are computed for each vehicle in a stability frame where the center of the frame is located at the center of gravity of the vehicle, axis y is aligned with the vehicle velocity vector V, and axis x is perpendicular to axis y, pointing to the right wing, which in the two-dimensional case lies in the vehicle-carried local level north–east plane. The isochronal contours for each aircraft may then be transformed by conventional methods into a single local level frame with a common origin.

Now consider a two-parameter (T, φ) grid for developing the isochronal contours for the system: for example $T = 0, 1, 2, \ldots, 10$ and angle $\varphi = 0, 5, 10, \ldots, 360$. For each $i = 1, 2, \ldots, n$ and each grid point, compute angle $\psi_i(T, \varphi)$ by Eq. (7) and the fact that $D = TV$. Using this family of ψ_i corresponding to the grid points, compute each contour set using Eqs. (3–6). Move the coordinate center of system $(X_1, Y_1)_i$ to the coordinate center specified by system (8, 9) as

$$X_{2i} = X_{1i} - x_i, \qquad Y_{2i} = Y_{1i} - y_i \tag{12}$$

Finally, turn Eq. (12) on the heading angle ω_i to obtain

$$x_i = X_{2i}\cos\omega_i + Y_{2i}\sin\omega_i \tag{13}$$

$$y_i = -X_{2i}\sin\omega_i + Y_{2i}\cos\omega_i \tag{14}$$

Assume that e is the lethal radius of the pursuer. For all values of T (on the time grid), the following inequalities are bounds on the pursuer multiple-evader problem:

$$|x_1 - x_j| \le e, \qquad |y_1 - y_j| \le e \tag{15}$$

where x_1, y_1 is the pursuer and $x_j, y_j, j = 2, 3, \ldots, N$ are the evaders.

When relation (15) holds, evader j cannot escape, even for the best evasion strategy. Relation (15) may be used to create constraints on a mathematical programming formulation of the single-evader/multiple-pursuer problem. For instance, a zero-failure policy

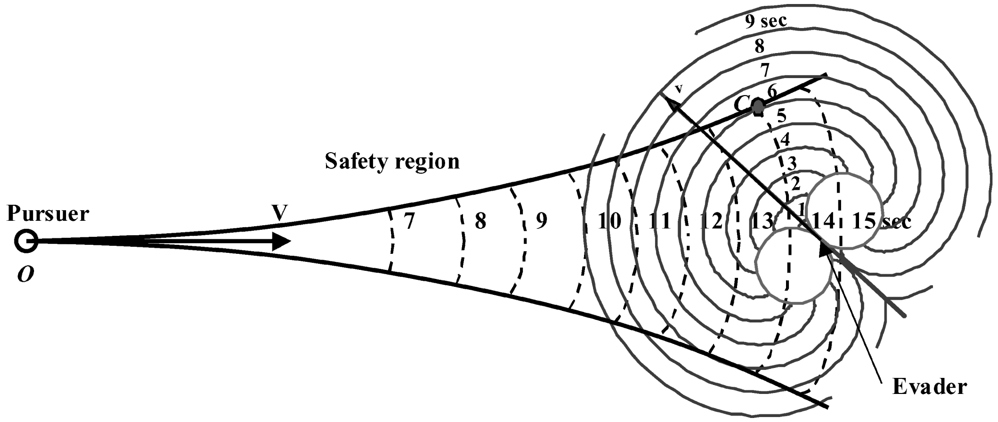

Fig. 8 Isochronal contours of pursuer and evader in direct attack.

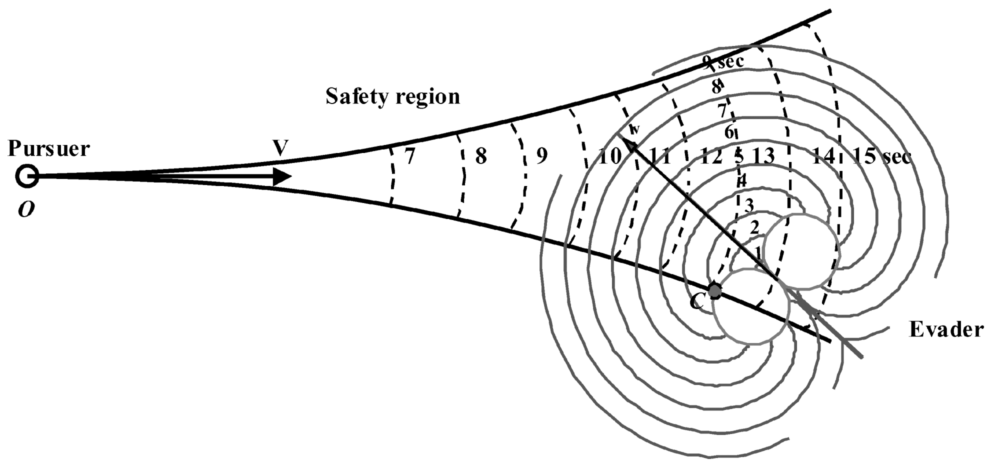

Fig. 9 Isochronal contours of pursuer and evader in leading attack.

would mean that only those evaders incapable of escape should be pursued. Hence, a constraint would be

$$\|[x_1(t), y_1(t)] - [x_j(t), y_j(t)]\| \leq e \qquad \text{for all} \quad t \in T \qquad (16)$$

where $\| \; \|$ denotes the 2-norm of the argument. Another possibility is to use relation (16) as a penalty term in the cost function. Of course, as events unfold, some evaders may become better candidates over time. Therefore it would be sensible to solve the mathematical programming problem recursively at discrete time intervals so that the target selection may adapt based on actual evader behavior.

V. Conclusions

Single-pursuer/multiple-evader problems have been around a long time. However, because of the limitations of game theory, satisfactory methods of treating this problem have been scarce. In this work, we presented a method that parameterizes the problem into a geometrically motivated bound. The bounds are constructed based on families of isochronal contours for both pursuer and evader. The interaction of pursuer and evader may then be captured algebraically as a bound or constraint on an optimization problem. It is the authors' opinion that this approach will yield extremely fast solution methods with fairly good performance. To obtain high-performance programs, much work needs to be done on developing specific penalty functions. An interesting approach might be to specify a "best" direction for the pursuer based on the accumulated penalty. This direction could then be modified periodically as new measurements are made and new bounds computed.

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